Concepts in Engineering Mathematics: Lecture 39

Part IV: Vector Calculus Lecture 39
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Mathematical Time Line 16-21 CE

|1450 |1500 |1600 |1700 |1800 |1900 |1950

Leonardo Newton Maxwell

Copernicus Descartes Gauss Einstein

Bombelli Huygens d'Alembert Riemann

Fermat Euler Cauchy Hilbert

Mersenne Johann Bernoulli Poncare

Galileo Jacob Bernoulli Weierstrass

Daniel Bernoulli Heaviside

Ben Franklin Abe Lincoln

- Review of last few lectures: Basic definitions
  - *Field*: i.e., Scalar & vector fields are functions of more than one variable
  - "Del:" $\nabla \equiv [\partial_x, \partial_y, \partial_z]^T$
  - Gradient: $\nabla\phi(x,y,z) \equiv [\partial_x\phi,\ \partial_y\phi,\ \partial_z\phi]^T$
- *Helmholtz Theorem:*
  Every vector field $\boldsymbol{V}(x,y,z)$ may be uniquely decomposed into *compressible* & *rotational* parts

$$\boldsymbol{V}(x,y,z) = -\nabla\phi(x,y,z) + \nabla\times\boldsymbol{A}(x,y,z)$$

- Scalar part $\nabla\phi$ is *compressible* ($\nabla\phi = 0$ is *incompressible*)
- Vector part $\nabla\times\mathbf{A}$ is *rotational* ($\nabla\times\boldsymbol{A} = 0$ is *irrotational*)
- Key vector identities: $\nabla\times\nabla\phi = 0$; $\nabla\cdot\nabla\times\mathbf{A} = 0$
- Definitions of Divergence, Curl & Maxwell's Eqs;
- Closure: Fundamental Theorems of Integral Calculus

Gradient: $\boldsymbol{E} = \nabla\phi(x, y, z)$ 39.14.3

- Definition: $\mathbb{R}^1 \underset{\nabla}{\mapsto} \mathbb{R}^3$

$$\mathbf{E}(x, y, z) = [\partial_x,\ \partial_y,\ \partial_z]^T\ \phi(x, y, z) = \left[\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right]^T_{x,y,z}$$

- $\mathbf{E} \perp$ plane tangent at $\phi(x, y, z) = \phi(x_0, y_0, z_0)$
- Unit vector in direction of $\mathbf{E}$ is $\hat{\mathbf{n}} = \frac{\mathbf{E}}{||\mathbf{E}||}$, along *isocline*
- Basic definition

$$\nabla\phi(x, y, z) \equiv \lim_{|\mathcal{S}|\to 0} \left\{ \frac{\iiint \phi(x, y, z)\, \hat{\mathbf{n}}\ dA}{|\mathcal{S}|} \right\}$$

$\hat{\mathbf{n}}$ is a unit vector in the direction of the gradient
$\mathcal{S}$ is the surface area centered at (x, y, z)

Divergence: $\nabla \cdot \boldsymbol{D} = \rho$ 39.14.4a

- Definition: $\mathbb{R}^3 \underset{\nabla\cdot}{\mapsto} \mathbb{R}^1$

$$\nabla \cdot \mathbf{D} \equiv [\partial_x,\, \partial_y,\, \partial_z] \cdot \mathbf{D} = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] = \rho(x,y,z)$$

- Examples:
  - Voltage about a point charge Q [SI Units of Coulombs]

$$\phi(x,y,z) = \frac{Q}{\epsilon_0 \sqrt{x^2+y^2+z^2}} = \frac{Q}{\epsilon_0 R}$$

    ϕ [Volts]; $Q = $ [C]; Free space ϵ_0 *permittivity* (μ_0 *permeability*)
  - *Electric Displacement* (flux density) around a point charge ($\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$)

$$\mathbf{D} \equiv -\nabla \phi(R) = -Q \nabla \left\{\frac{1}{R}\right\} = -Q\,\delta(R)$$

Divergence: The integral definition

- Surface integral definition of *incompressible* vector field

$$\nabla \cdot \mathbf{D} \equiv \lim_{|\mathcal{S}| \to 0} \left\{ \frac{\iint_{\mathcal{S}} \mathbf{D} \cdot \hat{\mathbf{n}} \, dA}{|\mathcal{V}|} \right\} = \rho(x, y, z)$$

$\mathcal{S}$ must be a closed surface
$\hat{\mathbf{n}}$ is the unit vector in the direction of the gradient

  - $\hat{\mathbf{n}} \cdot \mathbf{D} \perp$ surface differential dA

Divergence: Gauss' Law

39.14.4c

- General case of a *Compressible* vector field
- Volume integral over charge density $\rho(x, y, z)$ is total charge enclosed Q_{enc}
- $$\iiint_{\mathcal{V}} \nabla \cdot \boldsymbol{D} \, dV = \iint_{\mathcal{S}} \boldsymbol{D} \cdot \hat{\boldsymbol{n}} \, dA = Q_{enc}$$
- Examples
  - When the vector field is *incompressible*
    - $\rho(x, y, z) = 0$ [C/m^3] over enclosed volume
    - Surface integral is zero ($Q_{enc} = 0$)
  - Unit point charge: $D = \delta(R)$ [C/m^2]

Curl: $\nabla \times \mathbf{H} = \mathbf{I}$ [amps/m^2] 39.14.5a

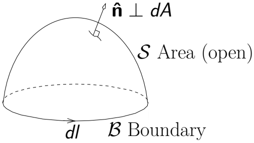
- Definition: $\mathbb{R}^3 \underset{\nabla\times}{\mapsto} \mathbb{R}^3$

$$\nabla \times \mathbf{H} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & H_z \end{vmatrix} = \mathbf{I}$$

- Examples:
  - Maxwell's equations: $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$, $\nabla \times \mathbf{H} = \sigma \mathbf{E} + \dot{\mathbf{D}}$,
  - $\mathbf{H} = -y\hat{x} + x\hat{y}$ then $\nabla \times \mathbf{H} = 2\hat{z}$ constant *irrotational*
  - $\mathbf{H} = 0\hat{x} + 0\hat{y} + z^2\hat{z}$ then $\nabla \times \mathbf{H} = \mathbf{0}$ is *irrotational*

Curl: Stokes Law

- Surface integral definition of $\nabla\times\mathbf{H} = \mathbf{I}$ $(\boldsymbol{I} \perp$ rotation plane of $\boldsymbol{H})$



$$\nabla\times\mathbf{H} \equiv \lim_{|\mathcal{S}|\to 0} \left\{ \frac{\iint_{\mathcal{S}} \hat{\mathbf{n}} \times \mathbf{H}\, dA}{|\mathcal{S}|} \right\} \quad (1)$$

$$\mathcal{I}_{enc} = \iint (\nabla\times\mathbf{H}) \cdot \hat{\boldsymbol{n}}\, dA = \oint_{\mathcal{B}} \mathbf{H} \cdot d\boldsymbol{l} \quad (2)$$

- Eq. (1): $\mathcal{S}$ must be an *open surface*
  with closed boundary $\mathcal{B}$
  $\hat{\mathbf{n}}$ is the unit vector $\perp$ to dA
  $\boldsymbol{H}\times\hat{\boldsymbol{n}} \in$ Tangent plane of A (i.e., $\perp \hat{\boldsymbol{n}}$)
- Eq. (2): Stokes Law: Line integral of $\boldsymbol{H}$ along $\mathcal{B}$ is total current $\mathcal{I}_{enc}$

Closure: Properties of fields of Maxwell's Equations 39.14.6

The variables have the following names and defining equations:

Symbol	Equation	Name	Units
$\mathbf{E}$	$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$	Electric Field strength	[Volts/m]
$\mathbf{D}$	$\nabla \cdot \mathbf{D} = \rho$	Electric Displacement (flux density)	[Col/m^2]
$\mathbf{H}$	$\nabla \times \mathbf{H} = \dot{\mathbf{D}}$	Magnetic Field strength	[Amps/m]
$\mathbf{B}$	$\nabla \cdot \mathbf{B} = 0$	Magnetic Induction (flux density)	[Weber/m^2]

In vacuo $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$, $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ [m/s], $r_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377$ [Ω].

Closure: Summary of vector field properties 39.14.7

- Notation:

$$\boldsymbol{v}(x,y,z) = -\nabla\phi(x,y,z) + \nabla\times\boldsymbol{w}(x,y,z)$$

- Vector identities:

$$\nabla\times\nabla\phi = 0; \;\; \nabla\cdot\nabla\times\mathbf{w} = 0$$

Field type	Generator:	Test (on $\boldsymbol{v}$):
Irrotational	$\boldsymbol{v} = \nabla\phi$	$\nabla\times\boldsymbol{v} = 0$
Rotational	$\boldsymbol{v} = \nabla\times\boldsymbol{w}$	$\nabla\times\boldsymbol{v} = \boldsymbol{J}$
Incompressible	$\boldsymbol{v} = \nabla\times\boldsymbol{w}$	$\nabla\cdot\boldsymbol{v} = 0$
Compressible	$\boldsymbol{v} = \nabla\phi$	$\nabla\cdot\boldsymbol{v} = \rho$

- Source density terms: Current: $\boldsymbol{J}(x,y,z)$, Charge: $\rho(x,y,z)$
  - Examples: $\nabla\times\boldsymbol{H} = \dot{\boldsymbol{D}}(x,y,z)$, $\nabla\cdot D = \rho(x,y,z)$

1. $f(x) \in \mathbb{R}$ (Leibniz Integral Rule): $F(x) = F(a) + \int_a^x f(x)dx$
2. $f(s) \in \mathbb{C}$ (Cauchy's formula): $F(s) = F(a) + \int_a^s f(\zeta)d\zeta$
   –When integral is independent of path, $F(s) \in \mathbb{C}$ obeys CR conditions
   –Contour integration inverts causal Laplace transforms
3. $\boldsymbol{F} \in \mathbb{R}^3$ (Helmholtz Formula): $\boldsymbol{F}(x, y, z) = -\nabla\phi(x, y, z) + \nabla \times \boldsymbol{A}(x, y, z)$
   –Decompose $\boldsymbol{F}(x, y, z)$ as *compressible* and *rotational*
4. Gauss' Law (Divergence Theorem): $Q_{enc} = \iiint \nabla \cdot \boldsymbol{D}\, dV = \iint_{\mathcal{S}} \boldsymbol{D} \cdot \hat{\boldsymbol{n}}\, dA$
   –Surface integral describes enclosed compressible sources
5. Stokes' Law (Curl Theorem): $\mathcal{I}_{enc} = \iint (\nabla \times \mathbf{H}) \cdot \hat{\boldsymbol{n}}\, dA = \oint_{\mathcal{B}} \mathbf{H} \cdot d\boldsymbol{l}$
   –Boundary vector line integral describes enclosed rotational sources
6. Green's Theorem . . . Two-port boundary conditions
   –*Reciprocity* property (*Theory of Sound*, Rayleigh, J.W.S., 1896)

Closure: Quasistatic (QS) approximation 39.14.9

- Definition: $ka \ll 1$ where a is the size of object, $\lambda = c/f$ wavelength
- This is equivalent to $a \ll \lambda$ or
- $\omega \ll c/a$ which is a low-frequency approximation
- The QS approximation is widely used, but infrequently identified.
- All *lumpted parameter models* (inductors, capacitors) are based on QS approximation as the lead term in a Taylor series approximation.